Appendix II
Method to Determine the Coefficients by Use of the Least-Square Method

The minimization of (26) is attained by minimizing both the real and imaginary parts of (26). Setting

$$g \equiv \mathrm{Im}\{S_{RW}\} \qquad f \equiv \mathrm{Re}\{S_{RW}\}$$

we have the minimization conditions

$$\frac{\partial f}{\partial G_0}=0, \frac{\partial f}{\partial G_\omega}=0, \frac{\partial f}{\partial G_V}=0, \frac{\partial f}{\partial G_C}=0, \frac{\partial f}{\partial G_{\omega 2}}=0 \qquad \text{(A3)}$$

$$\frac{\partial g}{\partial B_0}=0, \frac{\partial g}{\partial B_\omega}=0, \frac{\partial g}{\partial B_V}=0, \frac{\partial g}{\partial B_C}=0, \frac{\partial g}{\partial B_{\omega 2}}=0. \qquad \text{(A4)}$$

We find the coefficients of (4) by solving the simultaneous equations (A3) and (A4).

References

[1] D R. Hamilton, J. K. Knipp, and J. B. H. Kuper, *Klystron and Microwave Triodes*. New York: McGraw-Hill, Radiation Laboratory Series, 1948, pp. 404–440.
[2] J. R. Pierce and W. G. Shepherd, "Reflex oscillators," *Bell Syst. Tech. J.*, vol. 26, no. 3, pp. 460–681, 1947.
[3] M. Nakajima and J. Ikenoue, "Locking phenomena in microwave oscillator circuits," *Int. J. Electron.*, vol. 44, pp. 465–472, 1978.
[4] T. Hirota, M. Nakajima, and J. Ikenoue, "A simple equivalent circuit of a microstrip oscillator allowing for nonlinearity," *Int. J. Electron.*, vol. 48, pp. 427–434, 1980.

Single-Mode Fiber Design for Minimum Dispersion

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***Abstract*—The value of the radius of the core of a single-mode step-index optical fiber for minimum dispersion is calculated with the normalized frequency in the range $1.0 \leqslant V \leqslant 2.5$, using the approximation for the eigenvalue U proposed by Miyagi and Nishida [1]. This calculation is made by solving the total dispersion equation for the core radius when the wavelength assigned is assumed to be that necessary for minimum total dispersion. The computational procedure presented is simple enough to be accomplished on a programmable calculator or microcomputer. This work makes possible the characterization, with reasonable precision, of the ideal fiber that should be used with the available optical source.**

I. Introduction

The bandwidth for single-mode optical fibers is maximum when operation of the system takes place at the wavelength for minimum total dispersion $\hat{\lambda}$. Theoretical research concerning dispersion in monomodal step-index optical fibers has been based on the assumed prior knowledge of the core radius and of the materials that constitute the core and the cladding, so that the wavelength $\hat{\lambda}$ can be found. Since the wavelength is established by the characteristics of the known fiber, the next step is to search for the corresponding optical source.

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In previous publications [2], [3], the exact characteristic equations and numerical methods for differentiation and interpolation for calculating the value of $\hat{\lambda}$ for monomodal step-index optical fibers have been used. The results thus obtained were compared with those that were arrived at utilizing asymptotic formulas [4]–[7]. In spite of the excellent results obtained, even in the asymptotic limit, and having made possible the extension of the analysis that had been developed to other cases, the large quantity of calculations required the availability of medium to large computer systems. When such systems are not available, some approximate methods, with acceptable precision in these circumstances, allow the implementation of programs to calculate the value of $\hat{\lambda}$ using programmable calculators or microcomputer systems.

Utilizing the total dispersion formula established by South [7], derived with the objective of calculating $\hat{\lambda}$, together with the approximate formulation for the eigenfunction U proposed by Miyagi and Nishida [1], we prepared some programs for the TI-59 programmable calculator that make possible the design of monomodal step-index optical fibers. With the utilization of these programs, we obtain the value of the radius of the core, within the normalized frequency range $1.0 \leqslant V \leqslant 2.5$, for any value of the wavelength assumed to be that for minimum dispersion. In other words, we start with knowledge of the materials that will constitute the core and the cladding and with the value of the wavelength, and subsequently calculate the value of the core radius for which maximum information transfer will occur. Thus we have the possibility of characterizing with reasonable precision [8] the ideal optical fiber for use with the available source.

In Section II, we present the equations used, while in Section III, we will describe the computational methods implemented. In Section IV, we present some values of the core radii $\hat{a}$ for information transmission at minimum dispersion and some curves obtained for hypothetical fibers.

II. Formulation of the Problem

The value of the wavelength for minimum total dispersion $\hat{\lambda}$ depends on: a) the physical characteristics of the materials that constitute the core and the cladding, b) the core radius, and c) the propagation constant of the dominant HE_{11} mode and some of its derivatives. This value is calculated for the core radius a, with a predetermined value (for a known fiber) by solving the total dispersion equation [7]

$$D_T(a) = -\frac{\lambda}{cn_e}\left[(1-b)\nu_2 + b\nu_1 + 2b'\phi + \frac{1}{2}b''\theta - \frac{1}{n_e^2}\left(n_2n_2' + b\phi + \frac{1}{2}b'\theta\right)^2\right]\Big|_{\lambda=\hat{\lambda}} = 0 \qquad (1)$$

where c is the phase velocity in a vacuum, λ is the wavelength in free space

$$\nu_j = n_j n_j'' + (n_j')^2, \qquad j=1,2 \qquad (2a)$$

$$\phi = n_1n_1' - n_2n_2' \qquad (2b)$$

$$\theta = n_1^2 - n_2^2, \qquad n_e^2 = n_2^2 + b\theta. \qquad (2c)$$

The primes and double primes in (1) and (2) represent differentiations with respect to the wavelength λ. In (2a)–(2c), n_1 and n_2 represent the refractive indices of the core and cladding, respectively. In this paper, we will assume that the wavelength dependence of the refractive indices is given by the three-term Sellmeier equation

$$n_j^2 = 1 + \sum_{i=1}^{3} \frac{A_i \lambda^2}{\lambda^2 - l_i^2}, \qquad j = 1,2 \tag{3}$$

where each (A_i, l_i) pair is related to the optical properties of the materials involved. The derivatives of the refractive indices necessary for the calculation of the parameters described by (2a)–(2b) are given by the relations

$$n_j' = -\frac{1}{n_j} \sum_{i=1}^{3} \frac{A_i l_i^2 \lambda}{\left(\lambda^2 - l_i^2\right)^2}, \qquad j = 1,2. \tag{4}$$

$$n_j'' = \frac{1}{n_j}\left[-\left(n_j'\right)^2 + \sum_{i=1}^{3} \frac{A_i l_i^2 \left(3\lambda^2 + l_i^2\right)}{\left(\lambda^2 - l_i^2\right)^3}\right]$$

The parameter b in (1) is the normalized propagation constant for the dominant mode, given by

$$b = 1 - \frac{U^2}{V^2} \tag{5}$$

where U is one of the eigenfunctions of the characteristic equation and V is the normalized frequency given, as a function of the core radius, by

$$V = \frac{2\pi a}{\lambda} \theta^{1/2}. \tag{6}$$

The approximate formula for the parameter U that we will adopt in this work was proposed by Miyagi and Nishida [1] and is given by

$$U = \frac{U_\infty V}{V+1}\left[1 - \frac{1}{6}\frac{U_\infty^2}{(V+1)^3} - \frac{1}{20}\frac{U_\infty^4}{(V+1)^5}\right] \tag{7}$$

where $U_\infty = 2.40483$.

Substituting (7) in (5), we have

$$b = 1 - \frac{U_\infty^2}{(V+1)^2}\left[1 - \frac{1}{6}\frac{U_\infty^2}{(V+1)^3} - \frac{1}{20}\frac{U_\infty^4}{(V+1)^5}\right]^2. \tag{8}$$

Using (8), the derivatives b' and b'' that are encountered in (1) are calculated from

$$b' = -\frac{V(1-b)(2-\gamma)A}{V+1} \tag{9}$$

$$b'' = b'\left[\frac{b'}{(1-b)(2-\gamma)^2}\left(4\gamma + \frac{U_\infty^5}{(1-b)^{1/2}(V+1)^6}\right) - \left(\frac{1}{A}\left[\frac{\nu_1 - \nu_2}{\theta} - \left(\frac{\phi}{\theta}\right)^2\right] + \frac{2}{\lambda} + \frac{VA(\gamma - 6)}{2(V+1)}\right)\right] \tag{10}$$

where

$$\gamma = \frac{1}{U_\infty (1-b)^{1/2}}\left[\left(\frac{U_\infty}{V+1}\right)^4 + \frac{1}{2}\left(\frac{U_\infty}{V+1}\right)^6\right] \tag{11}$$

and

$$A = \frac{1}{\lambda} - \frac{\phi}{\theta}. \tag{12}$$

Instead of solving (1) to find $\lambda = \hat{\lambda}$, we choose a value of λ (assumed to be the wavelength of the available light source) and solve (1) for $a = \hat{a}$ where $\hat{a}$ is the core radius for which there will be minimum total dispersion for operation at $\hat{\lambda}$. Thus we are solving the equation

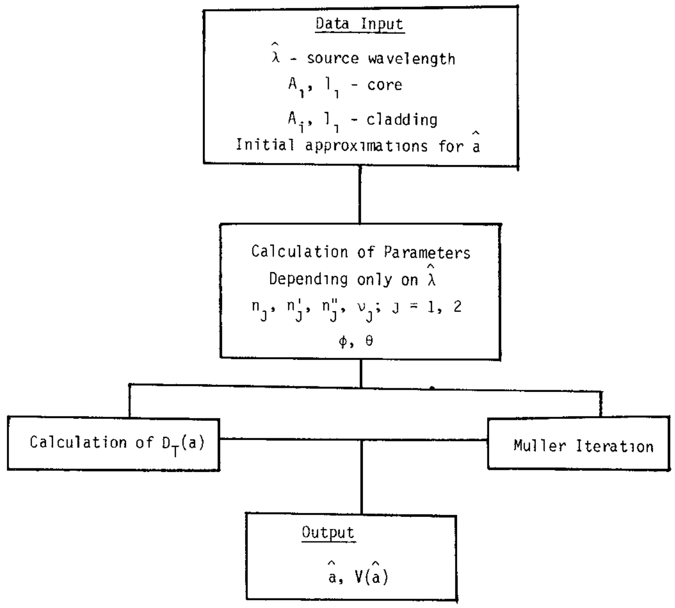

Fig 1 Main elements of the calculation process

$$D_T(\hat{\lambda})|_{a=\hat{a}} = 0, \qquad \hat{\lambda} = \text{constant}. \tag{13}$$

In Section III, the computational procedures necessary for solving (13) on small computer systems (or programmable calculators) will be outlined.

III. Computational Procedures

The process described above can be implemented on a programmable calculator or a small computer system using the approach to be described in this section and summarized in Fig. 1. Using the TI-59, 710 programming steps were used.

Part 1

The program requires the operating wavelength $\hat{\lambda}$, the values of the coefficients for the three-term Sellmeier equation for both the core and the cladding, and three initial approximations for the root $\hat{a}$ of (13), necessary for the initialization of the iterative process employed below.

Part 2

Here, the values of the parameters that depend only on the value of the wavelength $\hat{\lambda}$ are calculated. These parameters, obtained from (2)–(4), are constant during the execution of the program.

Part 3

This procedure consists of calculation of the parameters that depend on the core radius. This dependence is via the normalized frequency V (see (6)). The parameters calculated in this segment of the program are given by (5) and (7)–(12). Having made these calculations and having those results obtained in Part 2, we are able to calculate, using a subroutine, the values of $D_T(a)$ necessary for the iterative process described below.

Part 4

In Part 4, Muller iteration [9] is used to solve (13). This iterative process, besides presenting an almost quadratic convergence, doesn't require that the derivative of $D_T(a)$ be available. This procedure uses three initial approximations for $\hat{a}$ which can be obtained using bisection or a similar procedure. A subprogram can be used to calculate these approximations based on the value of the wavelength $\hat{\lambda}$ and on the coefficients of the three-term Sellmeier equations for the core and cladding materials. This subprogram calculates the initial value of the radius a_i given by the equation

$$a_i = \frac{\lambda}{2\pi\theta^{1/2}} \tag{14}$$

obtained by setting $V=1$ in (6), and with the increment set to some convenient value such as $\Delta V = 0.05$. Thus a search is made for the interval where $D_T(a)$ changes signs. This process is confined to the range $1.0 \leqslant V \leqslant 2.5$. The lower limit is established due to the behavior of b (given by (8)) for $V<1.0$ [2], while the upper limit is equal to the approximate limit of monomodal operation for the fiber. Once the interval is found, the initial approximations required by the main program are the limits of this interval, a_{i-2} and a_i, and the mean, $(a_{i-2}+a_i)/2 = a_{i-1}$.

Part 5

This part simply produces the values of $\hat{a}$ (solving (13)) and the values of $V(\hat{a})$ (from (6)). It is convenient also to include a procedure to produce D_T versus a for different values of $\hat{\lambda}$.

In the next section, we present the results obtained in the synthesis of some idealized fibers.

IV. Numerical Results

Using the procedures cited above, curves for total dispersion D_T as a function of fiber core radius a are presented for a wavelength $\hat{\lambda} = 1.55$ μm. This value of $\hat{\lambda}$ is the value for which the lowest loss has been found [10]. The curves for D_T versus a are presented in Figs. 2 and 3. The idealized fibers possess different concentrations of GeO_2 in SiO_2 as core materials and 100.0-percent fused SiO_2 (Fig. 2) and 100.0-percent quenched SiO_2 (Fig. 3) as cladding materials. The coefficients of the three-term Sellmeier equation for all the materials utilized in this work were obtained from [11]–[13]. In the figures presented, the initial value for the core radius a for each one of the curves corresponds to a value of the normalized frequency $V \cong 1.0$. The final value of the core radius corresponds to a value of $V \cong 2.5$ only for the curves labeled (A) in these figures.

Some values of $\hat{a}$ and $V(\hat{a})$ at constant values of $\hat{\lambda}$ were calculated for the same fibers considered in Figs. 2 and 3. The results, obtained to a precision of 10^{-5} for $\hat{a}$, are presented in Tables I and II. The average computational time necessary for calculating each one of the pairs of values $[\hat{a}, V(\hat{a})]$ using the TI-59 programmable calculator was 4 min. Interested readers may contact one of the authors for information on the availability of the TI-59 program listing.

Besides the synthesis programs described in Section III, a procedure for calculating the waveguide dispersion D_W as a function of core radius a for fixed values of $\hat{\lambda}$ was developed. The equations for D_W based on the Miyagi–Nishida approximation [1] may be found in the Appendix. At this point, it is important to note some distinctions between D_T and D_W.

According to [6], waveguide dispersion D_W is calculated by eliminating dispersive effects caused by the core and cladding materials. Obviously, these dispersive effects are due to the dependence of the refractive indices on the wavelength. In particular, the refractive index derivatives are strongly wavelength dependent. Thus if the first and second derivatives of n_1 and n_2 in (1) are eliminated, we have the equation for the waveguide dispersion (see (A1) in the Appendix). Since in monomodal operation a significant part of the dominant mode power propagates via the cladding, one chooses as values of n_1 and n_2 for the calculation of D_W the constant values they have at the wavelength of minimum dispersion λ_{mc} for the cladding material. It is important to note that the value of λ_{mc} does not cause the total

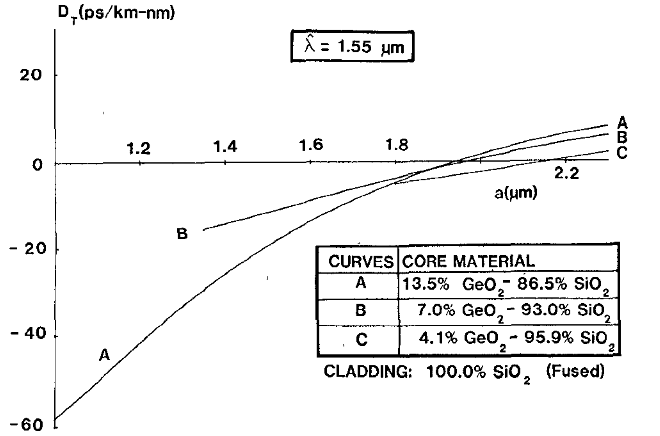

Fig. 2. The total dispersion D_T is shown as a function of the core radius a in microns, for different core materials and fused SiO_2 cladding. D_T is normally expressed in picoseconds per kilometer per nanometer or ps/(km·nm), indicated in the figure as, simply, ps/km·nm.

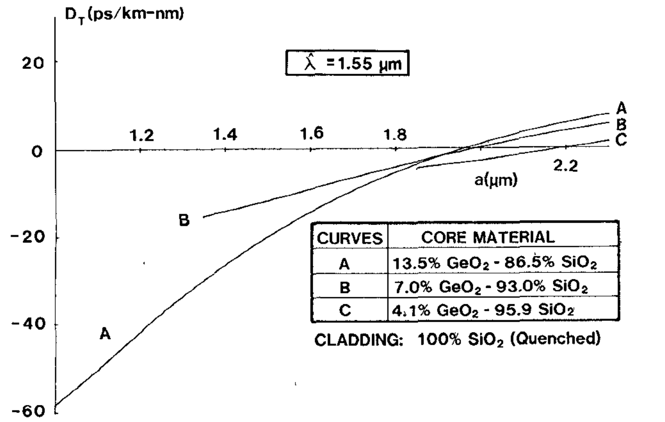

Fig. 3. The curves shown are similar to those of Fig. 2. However, the roots are shifted slightly since quenched SiO_2 cladding was used.

TABLE I
VALUES OF $\hat{a}$ AND $V(\hat{a})$ FOR FUSED SiO_2 CLADDING

$\hat{\lambda}$ (μm)	CORE MATERIAL					
	13.5% GeO_2 - 86.5% SiO_2		7.0% GeO_2 - 93.0% SiO_2		4.1% GeO_2 - 95.9% SiO_2	
	$\hat{a}$ (μm)	$V(\hat{a})$	$\hat{a}$ (μm)	$V(\hat{a})$	$\hat{a}$ (μm)	$V(\hat{a})$
1.45	2.1661	2.3618	2.3444	1.8298	2.5889	1.5120
1.50	2.0361	2.1507	2.1320	1.6149	2.3661	1.3363
1.55	1.9428	1.9906	1.9665	1.4475	2.1622	1.1823

TABLE II
VALUES OF $\hat{a}$ AND $V(\hat{a})$ FOR QUENCHED SiO_2 CLADDING

$\hat{\lambda}$ (μm)	CORE MATERIAL					
	13.5% GeO_2 - 86.5% SiO_2		7.0% GeO_2 - 93.0% SiO_2		4.1% GeO_2 - 95.9% SiO_2	
	$\hat{a}$ (μm)	$V(\hat{a})$	$\hat{a}$ (μm)	$V(\hat{a})$	$\hat{a}$ (μm)	$V(\hat{a})$
1.45	2.1905	2.3689	2.3787	1.8269	2.6468	1.5014
1.50	2.0568	2.1548	2.1584	1.6086	2.4122	1.3227
1.55	1.9614	1.9931	1.9872	1.4391	2.1955	1.1652

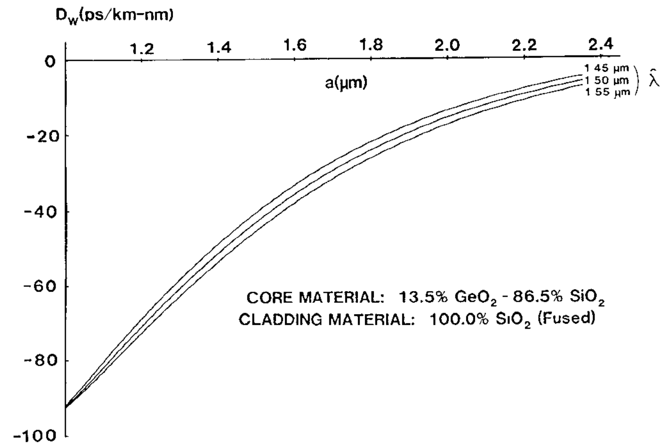

Fig. 4 The waveguide dispersion D_W is presented as a function of the core radius a for specific values of $\hat{\lambda}$. D_W is shown in ps/(km·nm) or, simply, ps/km-nm. The calculations were made for $n_1 = 1.46866$ and $n_2 = 1.44723$, obtained using the wavelength at which the cladding material dispersion is zero ($\lambda_{mc} = 1.2728$ μm)

dispersion to be zero (unless the core radius goes to infinity and the core and cladding have the same chemical composition). In the synthesis case, where the wavelength $\hat{\lambda}$ is known *a priori*, the derivatives of the core and cladding indices of refraction are nonzero. The choice of a fixed $\hat{\lambda}$ should guarantee the validity of (1). This means that the total dispersion is zero for $\lambda = \hat{\lambda}$ for some fixed radius a. Thus (13) is also satisfied. If, for a fixed core radius, we find a value of $\hat{\lambda}$ through the total dispersion equation (the analysis problem), for a constant value of $\hat{\lambda}$ assumed to satisfy (1), the ideal core radius of the fiber can be found (synthesis problem).

Having made these considerations, for simplicity, only fibers made of 13.5-percent GeO_2–86.5-percent SiO_2 for the core and 100.0-percent fused SiO_2 for the cladding are considered in Fig. 4. Here, some waveguide dispersion curves, D_W versus a, are presented for different values of $\hat{\lambda}$. Following the procedure outlined above, the value of the cladding refractive index n_2 is determined to be 1.44723, while the value of the core refractive index n_1 is found to be 1.46866, calculated using $\lambda_{mc} = 1.2728$ μm for 100.0-percent fused SiO_2 [2]. From the curves shown, we observe that, for fixed $\hat{\lambda}$, the magnitude of D_W decreases with increased core radius and that, for a given value of core radius, the magnitude of D_W increases with increasing $\hat{\lambda}$.

Due to the behavior of the normalized propagation constant b for values of $V < 1.0$ [2], attributed to the approximate formula used for the parameter U (such behavior affecting the b' and b'' derivatives), the method of synthesis used here does not permit us to find, for a single value of $\hat{\lambda}$, two values of the core radius. This fact, which can be seen through the analysis presented in [3], is the only limitation imposed on the theoretical development established here.

V. Conclusions

A method for the synthesis of single-mode step-index optical-fiber geometry for the normalized frequency range $1.0 \leqslant V \leqslant 2.5$ has been presented. The method, based on the Miyagi–Nishida approximation for the eigenvalue U [1], permits one to obtain the core radius $\hat{a}$ corresponding to a preestablished minimum total dispersion wavelength $\hat{\lambda}$ identified as that of an available optical source. This method permits the characterization with reasonable precision of the ideal fiber for use with a given source. All of the procedures discussed have been implemented on a small programmable calculator and are easily adaptable to a small computer.

Appendix

The expression for the waveguide dispersion, obtained from the total dispersion equation (1) with the first and second derivatives of the refractive indices set equal to zero for both the core and the clading, is given by

$$D_W = -\frac{\lambda}{c n_e}\left[\frac{1}{2}b''\theta - \frac{1}{n_e^2}\left(\frac{1}{2}b'\theta\right)^2\right] \tag{A1}$$

where

$$\theta = n_1^2 - n_2^2, \quad n_e^2 = n_2^2 + b\theta. \tag{A2}$$

The expression for the normalized propagation constant b continues to be that given by (8), while the expressions for the derivatives with respect to wavelength are given by

$$b' = -\frac{V(1-b)(2-\gamma)}{\lambda(V+1)} \tag{A3}$$

$$b'' = b'\left[\frac{b'}{(1-b)(2-\gamma)^2}\left(4\gamma + \frac{U_\infty^5}{(1-b)^{1/2}(V+1)^6}\right) - \left(\frac{2}{\lambda} + \frac{V(\gamma-6)}{2\lambda(V+1)}\right)\right]. \tag{A4}$$

The parameter γ is given by (11).

References

[1] M Miyagi and S. Nishida, "An approximate formula for describing dispersion properties of optical dielectric slab and fiber waveguides," *J. Opt Soc. Amer.*, vol. 69, pp. 291–293, Feb. 1979.

[2] P S M. Pires, "Analysis of dispersion in single-mode step-index optical fibers without the use of asymptotic expressions," M.S.E.E. thesis, State Univ of Campinas, Brazil, Aug 1980

[3] P S M. Pires, D. A. Rogers, E. J. Bochove, and R. F. Souza, "Prediction of laser wavelength for minimum total dispersion in single-mode step-index fibers," *IEEE Trans Microwave Theory Tech.*, vol MTT-30, pp 131–140, Feb. 1982.

[4] C T Chang, "Minimum dispersion in a single-mode step-index optical fiber," *Appl. Opt.*, vol. 18, pp. 2516–2522, July 1972.

[5] C. T. Chang, "Minimum dispersion at 1.55 μm for single-mode step-index fibres," *Electron. Lett*, vol 15, pp 765–767, Nov. 1979.

[6] D. Marcuse, "Interdependence of waveguide and material dispersion," *Appl. Opt.*, vol. 18, pp. 2930–2932, Sept. 1979.

[7] C R. South, "Total dispersion in step-index monomode fibers," *Electron. Lett*, vol. 15, pp. 394–395, June 1979.

[8] R A Sammut, "Analysis of approximations for the mode dispersion in monomode fibres," *Electron. Lett*, vol. 15, pp 590–591, Sept. 1979

[9] S. D. Conte, *Elementary Numerical Analysis*. New York: McGraw-Hill, 1965.

[10] T. Miya, Y. Tesunuma, T. Hosaka, and T. Miyashita, "Ultimate low-loss single mode fibres at 1 55 μm," *Electron. Lett.*, vol. 15, pp. 106–108, Feb. 1979

[11] J. W. Fleming, "Material dispersion in lightguide glasses," *Electron. Lett*, vol 14, pp. 326–329, May 1978.

[12] J. W. Fleming, "Material and mode dispersion in $GeO_2.B_2O_3$ SiO_2 glasses," *J. Amer. Ceram. Soc.*, vol. 59, pp. 503–507, Nov.–Dec. 1976.

[13] J H Malitson, "Interspecimen comparison of the refractive index of fused silica," *J. Opt. Soc Amer.*, vol. 55, pp 1205–1209, Oct. 1975.